Empirical Auctions Models II

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December 3, 2017

Introduction

- Structural Econometrics of Auction Data: why should we estimate the distribution of unobserved bidder valuations?
    - Main statistics of unobserved valuations
    - Effects of covariates on bidding behavior
    - Optimal reserve price
    - Expected revenues of the seller
    - Counterfactuals: auction design

Problem 1

- $\mathcal{N}$ potential bidders who are vying to purchase an object at a Vickrey auction within the **Independent Private-Value Paradigm (IPVP)**. (Non-cooperative game with incomplete information) (vs **Common Value Paradigm**)
- Given bidders valuations (**Unobserved**): $V_{(1:\mathcal{N})} > V_{(2:\mathcal{N})} > ... > V_{(\mathcal{N}:\mathcal{N})}$, bidder with the highest valuation $V_{(1:\mathcal{N})}$ wins the Vickrey auction and pays what his nearest opponent $V_{(2:\mathcal{N})}$ is willing to pay. (same CDF F_V^0)

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- At Vickrey auctions with reserve price, the dominant bidding strategy is to bid ones valuation, so bidder i's bid B_i is related to his valuation V_i according to

$$B_i = \beta(V_i) = V_i \;\; r \leq V_i$$

- r is the reserve price
- A collection of homogeneous objects is sold individually at a sequence of T different Vickrey auctions where $r = \$0.50$

Problem 1: distribution of valuations using winning bids

- Schema 1: at auction t, only the winning bid w_t as well as a measure of the number of potential competitors $\mathcal{N}_t$ is recorded $(\{(\mathcal{N}_t, w_t)\}_{t=1}^{T})$
- Estimate the distribution of unobserved valuations using the winning bids W. (see **"win.dat"**)

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- Relationship between observed bids and valuations

$$f_B^0(b) = f_V^0(v) \ , \ \ F_B^0(b) = F_V^0(v)$$

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- (for $r = 0$) The cumulative distribution function of the second-highest order statistic equals

$$F_W(w|\mathcal{N}_t) = \mathcal{N}_t F_V(w)^{\mathcal{N}_t - 1} - (\mathcal{N}_t - 1) F_V(w)^{\mathcal{N}_t}$$

following (Arnold, Balakrishnan, and Nagaraja (1992)) the distribution of any single order statistic $(k : \mathcal{N})$ has the distribution

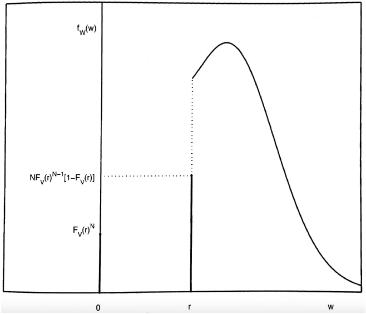
$$F_Y^{(k:\mathcal{N})}(s) = \frac{\mathcal{N}!}{(\mathcal{N} - k)!(k-1)!} \int_0^{F_Y(s)} u^{\mathcal{N}-k}(1-u)^{k-1} du$$

Problem 1: distribution of valuations using winning bids

- (for $r = 0$) the corresponding density function

$$f_W(w|\mathcal{N}_t) = \frac{F_W(w|\mathcal{N}_t)}{\partial w} = \mathcal{N}_t(\mathcal{N}_t - 1)F_V(w)^{\mathcal{N}_t - 2}[1 - F_V(w)]f_V(w)$$

- (for $r > 0$) Truncated distribution



- (Amemiya (1985)) The density function where $\{(\mathcal{N}_t, w_t)\}_{t=1}^{T}$ are available equals

$$f_{W_t}(w) = \frac{\mathcal{N}_t(\mathcal{N}_t - 1)F_V(w)^{\mathcal{N}_t - 2}[1 - F_V(w)]f_V(w)}{[1 - F_V(r)^{\mathcal{N}_t}]}$$

Problem 1: distribution of valuations using winning bids

- The likelihood function (L) is

$$L = \Pi_{t=1}^{T} \frac{\mathcal{N}_t(\mathcal{N}_t - 1)F_V(w)^{\mathcal{N}_t - 2}[1 - F_V(w)]f_V(w)}{[1 - F_V(r)^{\mathcal{N}_t}]}$$

- Assume that the distribution of heterogeneity is from the Weibull family (**show distribution**)

$$f_V(v;\theta) = \theta_1\theta_2 v^{\theta_2 - 1} exp(-\theta_1 v^{\theta_2}) \ \ v \geq 0, \theta_1 > 0, \theta_2 > 0$$

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- Estimate θ_1 and θ_1 by method of maximum likelihood
- Objective function (see **prog33slide1.m**)

$$min_\theta - log(L)$$

- Estimates $\hat{\theta}_1^{ML} = 0.9088$ and $\hat{\theta}_2^{ML} = 2.0145$
- Plot truncated $f_V(v;\theta)$
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Problem 1: Estimation of the distribution of winning bids

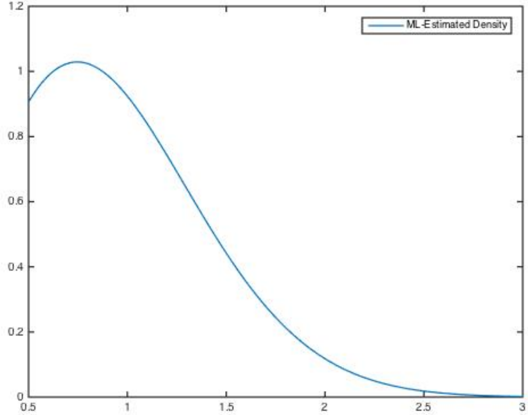

Figure: Truncated Density Function

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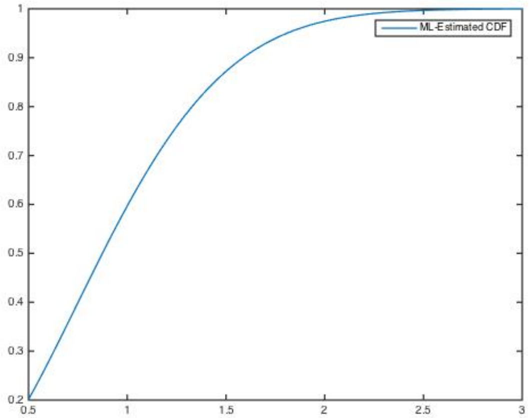


Figure: Truncated CDF

Problem 1: Nonparametric estimate of $F_V^0(v)$

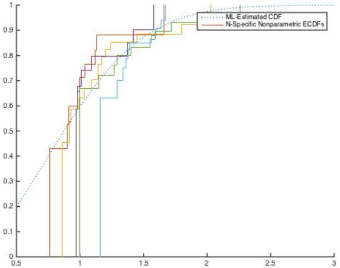
- Calculate an estimate of $F_V^0(v)$ for valuations that are above the reserve price.
- When $\mathcal{N}$ does not vary across auctions a np estimator of CDF of W equals

$$\hat{F}_W(w) = \frac{1}{T}\sum_{t=1}^{T} I(W_t \leq w_t)$$

- Given $\hat{F}_W(w)$ we can estimate $\hat{F}_V(v)$ solving

$$\hat{F}_W(w|\mathcal{N}_t) = \mathcal{N}_t \hat{F}_V(w)^{\mathcal{N}_t - 1} - (\mathcal{N}_t - 1)\hat{F}_V(w)^{\mathcal{N}_t}$$

- **"win.dat" (prog33fslide2.m)**: five estimates of $\hat{F}_V$ for different values of $\mathcal{N} = [5, 6, 7, 8, 9]$



Problem 1: Maximum likelihood estimate of the optimal reserve price ρ^*

- Test if the existing reserve price of \$0.50 is optimal
- (Riley and Samuelson (1981)) The expected PROFIT for the seller is

$$v_0 F_V(r)^{\mathcal{N}} + \mathcal{N} \int_r^{\bar{v}} [u f_V(u) + F_V(u) - 1] F_V(u)^{\mathcal{N}-1} du$$

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- Differentiating this wrt the reserve price r and equalling zero ($r \to \rho^*$)

$$\mathcal{N} v_0 F_V(r)^{\mathcal{N}-1} f_V(r) - \mathcal{N}[r f_V(r) + F_V(r) - 1] F_V(r)^{\mathcal{N}-1} = 0$$

- Optimal reserve price (ρ^*)

$$\rho^* = v_0 + \frac{[1 - F_V(\rho^*)]}{f_V(\rho^*)} \qquad \sqrt{T}(\hat{\rho} - \rho^0) \to \mathcal{N}(0, \Omega)$$

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- Is $\rho = 0.5$ optimal? (see **prog33jslide6.m**)

$$H_0 : \rho^* = 0.5 \quad H_1 : \rho^* \neq 0.5$$

  for $\alpha = 0.05$ we reject H_0 if

$$\frac{\rho^* - 0.5}{\sqrt{\hat{\Omega}}} > 1.96$$

- (when $v_0 = 0$): ρ^* is 0.7407 with a std of 0.0431, standard-normal test statistic is 5.5789: **REJECT** the hypothesis that an r of 0.50 is optimal

Problem 1: Covariates

- Covariates control for heterogeneity of objects (e.g. year, month) (Z)
- Density and cumulative distribution functions of V conditional on covariates (Z)

$$f^0_{V|Z}(v|z) \quad F^0_{V|Z}(v|z)$$

- Introduce covariate z_t in the probability density function (see **"win.dat"**)

$$f_V(v;\theta,Z_t) = exp(\theta_{10}+\theta_{11}z_t)\theta_2 v^{\theta_2-1}exp(-exp(\theta_{10}+\theta_{11}z_t)v^{\theta_2})$$

where $\theta_1 = exp(\theta_{10}+\theta_{11}z_t)$

- Maximum likelihood estimates (see **prog33hslide4.m**):

$$min_\theta - log(L)$$

$\theta_{10} = -0.1230$ $\theta_{11} = 0.0062$ $\theta_2 = 2.0138$, $log(L) = -5.6558$

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  $\theta_{10} = -0.1230$ $\theta_{11} = 0.0062$ $\theta_2 = 2.0138$, $log(L) = -5.6558$

- Test whether $\theta_{11} = 0$ using the likelihood ratio-test

$$H_0 : \theta_{11} = 0 \quad H_0 : \theta_{11} \neq 0$$

$$\text{Ratio-Test Statistic} \Rightarrow LR = -2*(log(L_0) - log(L_1)) \sim \chi^2_{\alpha,df}$$

  for $\alpha = 0.05$ and df equals Nb parameters H_1 - Nb parameters H_0 and equals 1
  $\Rightarrow \chi^2_{0.05,1} = 3.8414$, $LR = -2*(-5.6940+5.6558) = 0.0763$

- $LR < 3.8414 \Rightarrow$ we cannot reject H_0

Problem 2: distribution of valuations using submitted bids

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$$B_i = \beta(V_i) = V_i \;\; r \leq V_i$$

- r is the reserve price
- A collection of homogeneous objects is sold individually at a sequence of T different Vickrey auctions where $r = \$0.50$
- Schema 2: at auction t, only the bids submitted b_t, which equals $[b_{(1:n_t)}, b_{(2:n_t)}, ..., b_{(n_t:n_t)}]$ are available (see **"bids.dat"**)

Problem 2: distribution of valuations using submitted bids

- The probability density function where $\{(b_t, n_t)\}_{t=1}^{T}$ are available

$$g[b_{(n)}, n] = n! \Pi_{i=1}^{n} \frac{f_V[b_{(i:n)}]}{[1 - F_V(r)]}$$

which implies that the likelihood function is

$$L = \Pi_{t=1}^{T} n_t! \Pi_{i=1}^{n_t} \frac{f_V[b_{(i:n_t)}]}{[1 - F_V(r)]}$$

- Assume that the distribution of heterogeneity is from the Weibull family

$$f_V(v; \theta) = \theta_1 \theta_2 v^{\theta_2 - 1} exp(-\theta_1 v^{\theta_2}) \;\; v \geq 0, \theta_1 > 0, \theta_2 > 0$$

- Estimate θ_1 and θ_1 by method of maximum likelihood
- Objective function

$$min_{\theta} - log(L)$$

- (see **"bids.dat"**) Maximum Likelihood Estimates: $\hat{\theta}_1^{ML} = 0.9177$ and $\hat{\theta}_2^{ML} = 2.1138$

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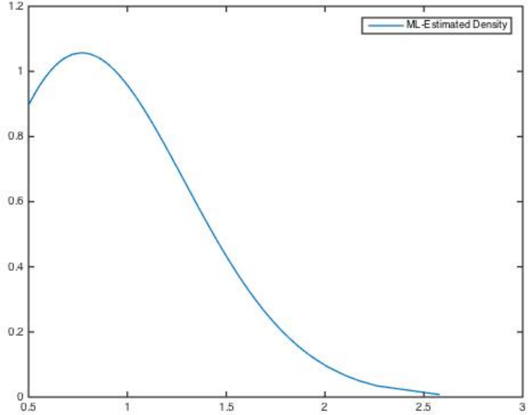

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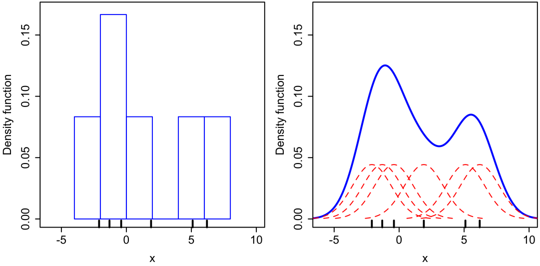
kernel-smoothed estimate of $f_V^0(v)$



- Kernel Estimator

$$\hat{f}_V(v) = \frac{1}{Th} \sum_{t=1}^{T} \kappa\left(\frac{V_t - v}{h}\right)$$

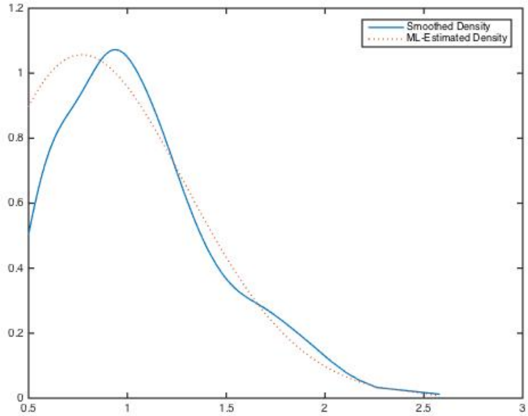
where

$$\kappa\left(\frac{V_t - v}{h}\right) = \frac{1}{\sqrt{2\pi}} exp\left[\frac{-(V_t - v)^2}{2h^2}\right]$$

h is the bandwidth (Silverman (1986) $1.06\sigma T^{-1/5}$)

Problem 2: distribution of valuations using submitted bids

kernel-smoothed estimate of $f_V^0(v)$



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